ON THE STABILITY OF GYROCOMPASSES

(OB USTOICHIVOSTI GIROKOMPASOV)

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In the present study, which is an extension of [1], we consider in the rigorous formulation (for the nonlinear case and without passing to precession theory) the stability of the motion of two-rotor gyrocompasses not having the properties of the Geckeler-Anschütz spatial gyrocompass.

The article gives the first integral of the equations of motion, which is used for obtaining sufficient conditions for the stability of the unperturbed motion of the system.

1. Let $Ox^{\circ}y^{\circ}z^{\circ}$ and $Oxyz$ be right-handed coordinate systems whose origins coincide with the point of suspension [2].

The attitude of the trihedron $Oxyz$ with respect to the trihedron $Ox^{\circ}y^{\circ}z^{\circ}$ will be defined by the angles α, β and γ. The direction cosines θ_s, ϑ_s and ψ_s between these trihedrons are defined by equations (1.1) of [1].

The motion of a two-rotor gyrocompass is described by the system of four equations (1.2) given in [1].

In the present study we shall assume that the restoring moment of the spring connection between the gyroscopes satisfies the condition

$$N = s \sin \delta \cos \delta \tag{1.1}$$

where s is the slope of the characteristic of the moment in question and δ is the angle by which the gyroscope axes deviate from the selected design value, which would correspond to $\varepsilon = \varepsilon_0$.

Thus

$$\delta = \varepsilon - \varepsilon_0 \tag{1.2}$$

Unlike [1], the present investigation will take the term $-2Id^2\varepsilon/dt^2$ into account in the equation describing the motion of the gyroscopes within the gyrosphere.

Using the same assumptions as in [1], the above system admits of a first integral, which is obtained in a way completely similar to [1].

This integral is of the form

$$V \equiv {}^1/_2Ap^2 + {}^1/_2Bq^2 + {}^1/_2Cr^2 + I\dot{\delta}^2 + {}^1/_2s \sin^2\delta - (F - mv^2 / R)\, l\psi_3 - mvl\Omega\vartheta_3 -$$
$$- [Ap\psi_1 + (Bq + \mathrm{H})\,\psi_2 + Cr\psi_3]\,\Omega - [Ap\vartheta_1 + (Bq + \mathrm{H})\,\vartheta_2 + Cr\vartheta_3]\, v / R = C_1 \tag{1.3}$$

2. The integral (1.3) can be used to obtain sufficient conditions for stability. The equilibrium positions of the system will be represented by the following values of the coordinates:

$$\alpha = 0, \qquad \beta = \beta^*, \qquad \gamma = 0, \qquad \delta = \delta^* \tag{2.1}$$

where β^* and δ^* satisfy the equations

$$(C - B)\,\{{}^1/_2\,[\Omega^2 - \omega^2] \sin 2\beta^* + \omega\,\Omega \cos 2\beta^*\} -$$
$$-2B' \cos(\varepsilon_0 + \delta^*)\,(\Omega \cos\beta^* - \omega \sin\beta^*) = -(F - mv\omega)\, l \sin\beta^* - mvl\Omega \cos\beta^* \tag{2.2}$$
$$-(\omega \cos\beta^* + \Omega \sin\beta^*)\, 2B' \sin(\varepsilon_0 + \delta^*) = s \sin\delta^* \cos\delta^* \qquad (\omega = v/R)$$

The motion defined by equations (2.1) will be considered as the unperturbed motion.

We now consider the perturbed motion

$$\alpha = x_1, \qquad \beta = \beta^* + x_2, \qquad \gamma = x_3, \qquad \delta = \delta^* + x_4 \tag{2.3}$$

Designating by V_0 the value of V when $x_s = 0$ and $\dot{x}_s = 0$ ($s = 1, 2, 3, 4$), we calculate the difference $V - V_0$. We have

$$V - V_0 = \sum_{i,\, j=1}^{4} b_{ij}\dot{x}_i\dot{x}_j + \sum_{k,\, l=1}^{4} c_{kl}x_kx_l + \ldots \tag{2.4}$$

where the symbol ... represents higher-order terms in x_s and $\dot{x}_s$.

Here (2.5)

$$b_{11} = {}^1/_2\,(B \sin^2\beta^* + C \cos^2\beta^*), \qquad b_{22} = {}^1/_2A, \qquad b_{33} = {}^1/_2B \qquad b_{44} = I$$
$$b_{12} = b_{21} = 0, \qquad b_{13} = b_{31} = {}^1/_2B \sin\beta^*, \qquad b_{14} = b_{41} = 0$$
$$b_{23} = b_{32} = 0, \qquad b_{24} = b_{42} = 0, \qquad b_{34} = b_{43} = 0$$

$$c_{11} = {}^1/_2\, \omega\, \{-\, mlR\,\Omega \sin \beta^* - A\omega + [B\, (\cos \beta^*\omega + \Omega \sin \beta^*) + \\ + 2B' \cos\, (\varepsilon_0 + \delta^*)] \cos \beta^* + C\, (\sin \beta^*\omega - \Omega \cos \beta^*) \sin \beta^*\}$$

$$c_{22} = {}^1/_2\, \{(C - B)\, [(\Omega \cos \beta^* - \sin \beta^*\omega)^2 - (\Omega \sin \beta^* + \cos \beta^*\omega)^2] + \\ + [(F - mv\omega)\, l + 2B' \cos\, (\varepsilon_0 + \delta^*)\, \omega] \cos \beta^* - \Omega\, [mvl - 2B' \cos\, (\varepsilon_0 + \delta^*)] \sin \beta^*\}$$

$$c_{33} = {}^1/_2\, [(C - A)\, (\Omega \cos \beta^* - \sin \beta^*\, \omega)^2 + (F - mv\omega)\, l \cos \beta^* - mvl\Omega \sin \beta^*]$$

$$c_{44} = {}^1/_2\, [s \cos 2\delta^* + 2B' \cos\, (\varepsilon_0 + \delta^*)\, (\Omega \sin \beta^* + \cos \beta^*\, \omega)]$$

$$c_{12} = c_{21} = 0, \qquad c_{14} = c_{41} = 0, \qquad c_{23} = c_{32} = 0, \qquad c_{34} = c_{43} = 0$$

$$c_{13} = c_{31} = {}^1/_2\, \omega [(C - A)\, (\sin \beta^*\, \omega - \Omega \cos \beta^*) - mlR\,\Omega]$$

$$c_{24} = c_{42} = {}^1/_2 2B' \sin\, (\varepsilon_0 + \delta^*)\, (\Omega \cos \beta^* - \sin \beta^*\, \omega)$$

Applying to the first quadratic form of expression (2.4) Sylvester's criterion, we obtain the always implemented inequality

$$B \sin^2 \beta^* + C \cos^2 \beta^* > 0, \qquad A > 0, \qquad BC \cos^2 \beta^* > 0, \qquad I > 0 \tag{2.6}$$

Applying Sylvester's criterion to the second quadratic form in formula (2.4), we find that it is positive-definite for sufficiently small values of x_s if the following inequalities are satisfied:

$$c_{11} > 0, \qquad c_{22} > 0, \qquad c_{11}c_{33} - c_{13}^2 > 0, \qquad c_{22}c_{44} - c_{24}^2 > 0 \tag{2.7}$$

The for $V - V_0$ will also be positive-definite if conditions (2.7) are satisfied.

Since its total derivative is identically equal to zero by virtue of the equations of perturbed motion, it follows that the unperturbed motion (2.1) will be stable in the Liapunov sense. It should be noted that equations (2.2) have the solution $\beta^* = 0$ if ε_0 satisfies the equation

$$\varepsilon_0 = \cos^{-1} \left[\frac{mlv}{2B'}(1 + \chi)\right] + \frac{1}{2} \sin^{-1} \frac{4B'v}{sR}\left[1 - \left(\frac{mlv}{2B'}(1 + \chi)\right)^2\right]^{1/2} \\ \left(\chi = \frac{(C - B)}{mlR}\right) \tag{2.8}$$

The value δ^* is determined by the equation [3]

$$2B' \cos\, (\varepsilon_0 + \delta^*) = mlv\, (1 + \chi) \tag{2.9}$$

Introducing the notation

$$p^2 = \frac{Fls}{[2B' \sin (\varepsilon_0 + \delta^*)]^2} \tag{2.10}$$

we find the following explicit expressions for the coefficients c_{kl}:

$$c_{11} = \frac{1}{2} ml \frac{v^2}{R}\left[1+\frac{(C-A)}{mlR}\right], \qquad c_{33} = \frac{1}{2}\left(F - m\frac{v^2}{R}\right) l \left[1+\frac{(C-A)\,\Omega^2}{(F-m\,v^2/R)l}\right]$$
$$c_{22} = \frac{1}{2} Fl\left[1+\chi\left(\frac{\Omega}{\nu}\right)^2\right], \qquad c_{44} = \frac{1}{2}\left[s\cos 2\delta^* + ml\frac{v^2}{R}(1+\chi)\right] \tag{2.11}$$
$$c_{13} = c_{31} = -\frac{1}{2} mlv\Omega\left[1+\frac{(C-A)}{mlR}\right], \qquad c_{24} = c_{42} = \frac{1}{2}\frac{\sqrt{Fls}\,\Omega}{p}$$

and the inequalities (2.7) assume the simple form

$$mlv\left[1+\frac{(C-A)}{mlR}\right] > 0, \qquad Fl\left[1+\chi\left(\frac{\Omega}{\nu}\right)^2\right] > 0, \qquad F - m\frac{v^2}{R} - mR\,\Omega^2 > 0$$
$$p^2\left[1+\chi\left(\frac{\Omega}{\nu}\right)^2\right]\left[\cos 2\delta^* + \frac{mlv^2}{sR}(1+\chi)\right] - \Omega^2 > 0 \tag{2.12}$$

3. The sufficient conditions for stability (2.12) permit degeneration to the case of precession theory. If in (2.12) we neglect the terms containing the quantities A, B and C as factors and if we set $\cos 2\delta^*$ equal to unity because the angle δ^* is so small, we obtain the following inequalities:

$$F - m\frac{v^2}{R} - mR\,\Omega^2 > 0, \qquad p_1^2 - \Omega^2 > 0 \qquad \left(p_1^2 = p^2\left(1+\frac{mlv^2}{sR}\right)\right) \tag{3.1}$$

Since, in addition, we always have $sR \gg mlv^2$, it follows that we may take $p_1 = p$. If we now set $F - mv^2/R \approx mg$, we obtain from (3.1) the inequalities

$$\Omega^2 - p^2 < 0, \qquad \Omega^2 - \nu^2 < 0 \qquad (\nu = \sqrt{g/R}) \tag{3.2}$$

previously established in [4]. These same inequalities are also obtained for the case of complete kinetic symmetry, when $A = B = C$.

To obtain necessary conditions for stability, we can consider the corresponding equations in the variations.

The precession formulation of this problem was considered in [4]. It was shown there that if

$$(\Omega^2 - p^2)(\Omega^2 - \nu^2) < 0 \tag{3.3}$$

then the unperturbed motion is unstable.

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